

On structure of homogenous Wick ideals in Wick \ast -algebras with braided coefficients

Vasyl Ostrovskiy^(a) Danil Proskurin^(b) Yurii Savchuk^(c)
Lyudmila Turowska^(d)

1 Introduction

In this paper we present some results on structure of Wick homogenous ideals of quadratic algebras allowing Wick ordering, shortly Wick algebras, introduced in [3]. Namely, let $\{T_{ij}^{kl}, i, j, k, l = 1, \dots, d\} \subset \mathbb{C}$ satisfy conditions $T_{ji}^{lk} = \overline{T_{ij}^{kl}}$, then Wick algebra $W(T)$ is generated by $a_i, a_i^*, i = 1, \dots, d$, satisfying commutation relations of the form

$$a_i^* a_j = \delta_{ij} 1 + \sum_{k,l=1}^d T_{ij}^{kl} a_l a_k^*, \quad i, j = 1, \dots, d. \quad (1)$$

Following [3] consider finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}\langle e_1, \dots, e_d \rangle$ and its formal dual $\mathcal{H}^* = \mathbb{C}\langle e_1^*, \dots, e_d^* \rangle$, where $\{e_i, i = 1, \dots, d\}$ form an orthonormal base of \mathcal{H} . Put $\mathcal{T}(\mathcal{H}, \mathcal{H}^*)$ to be the full tensor algebra over \mathcal{H} and \mathcal{H}^* , then

$$W(T) \simeq \mathcal{T}(\mathcal{H}, \mathcal{H}^*) / \langle e_i^* \otimes e_j - \sum_{k,l=1}^d T_{ij}^{kl} e_l \otimes e_k^* \rangle. \quad (2)$$

Note, that in this realisation the free algebra generated by $a_i, i = 1, \dots, d$ coincides with $\mathcal{T}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n}$.

The Fock representation of $W(T)$ is defined on $\mathcal{T}(\mathcal{H})$ by the rules

$$a_i^* \Omega = 0, \quad a_i e_{i_1} \otimes \dots \otimes e_{i_k} = e_i \otimes e_{i_1} \otimes \dots \otimes e_{i_k}, \quad i = 1, \dots, d,$$

the action of $a_i^*, i = 1, \dots, d$, on vectors other than Ω , is determined inductively using the commutation relation in $W(T)$. It was proved in [3] that there exists a unique sesquilinear form $\langle \cdot, \cdot \rangle_F$, called the *Fock scalar product*, on $\mathcal{T}(\mathcal{H})$,

^(a) Institute of Mathematics, NAS of Ukraine, Ukraine, vo@imath.kiev.ua

^(b) Kyiv Taras Shevchenko University, Ukraine, prosk@univ.kiev.ua

^(c) Leipzig University, Germany, Yuriy.Savchuk@math.uni-leipzig.de

^(d) Chalmers University of Technology and University of Gothenburg, Sweden, turowska@chalmers.se

such that the Fock representation becomes a $*$ -representation with respect to this form. It is defined in such a way that the subspaces $\mathcal{H}^{\otimes n}$ and $\mathcal{H}^{\otimes m}$ are orthogonal if $m \neq n$ and

$$\langle X, Y \rangle_F = \langle X, P_n Y \rangle, \quad X, Y \in \mathcal{H}^{\otimes n}.$$

where by $\langle \cdot, \cdot \rangle$ we denote the standard scalar product on $\mathcal{H}^{\otimes n}$ and $P_n: \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$ is an operator defined in the following way (see [3]): First we introduce an operator $T: \mathcal{H}^{\otimes 2} \rightarrow \mathcal{H}^{\otimes 2}$ given by

$$T e_k \otimes e_l = \sum_{i,j=1}^d T_{ik}^{lj} e_i \otimes e_j. \quad (3)$$

Note that T is self-adjoint with respect to the standard scalar product on $\mathcal{H}^{\otimes 2}$. Further, for any $n > 2$ consider the following extensions of T to $\mathcal{H}^{\otimes n}$:

$$T_i = \bigotimes_{k=1}^{i-1} \mathbf{1}_{\mathcal{H}} \otimes T \otimes \bigotimes_{k=i+2}^n \mathbf{1}_{\mathcal{H}}, \quad i = 1, \dots, n-1.$$

Then we set $P_0 = 1$, $P_1 = \mathbf{1}_{\mathcal{H}}$, $P_2 = \mathbf{1}_{\mathcal{H}^{\otimes 2}} + T$ and

$$P_n = (\mathbf{1}_{\mathcal{H}} \otimes P_{n-1}) R_n, \quad n \geq 3, \quad (4)$$

where

$$R_n: \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}, \quad R_n = \mathbf{1}_{\mathcal{H}^{\otimes n}} + T_1 + T_1 T_2 + \dots + T_1 T_2 \dots T_{n-1}.$$

Remark 1. The operators R_n , $n \geq 2$, are used to obtain explicit formulas for commutation relations between generators a_i^* , $i = 1, \dots, d$ and homogeneous polynomial in noncommutative variables a_1, \dots, a_d . Namely, by [6], for $X \in \mathcal{H}^n$ one has the following equality in $W(T)$ (here we use the canonical realisation)

$$e_i^* \otimes X = \mu_0(e_i^*)(R_n X + \sum_{k=1}^d T_1 T_2 \dots T_n (X \otimes e_k) \otimes e_k^*),$$

where $\mu_0(e_i^*): \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ is given by

$$\mu_0(e_i^*) e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_s} = \delta_{ii_1} e_{i_2} \otimes \dots \otimes e_{i_s}, \quad s \geq 1, \quad \mu_0(e_i^*) \Omega = 0.$$

This allows to determine explicitly the action of a_i^* in the Fock representation as follows

$$a_i^* X = \mu_0(e_i^*) R_n X, \quad X \in \mathcal{H}^{\otimes n}.$$

Positivity of the Fock scalar product means that $P_n \geq 0$ for all $n \geq 2$. In this case the Fock representation can be extended to a $*$ -representation of $W(T)$ on a Hilbert space, which is a completion of $\mathcal{T}(\mathcal{H}) / \bigoplus_{n \geq 2} \ker P_n$ with respect to the norm defined by the Fock scalar product. Sufficient conditions for positivity

of family $\{P_n, n \geq 2\}$ can be found in [1, 2, 3]. For instance if T is *braided*, i.e. $T_1 T_2 T_1 = T_2 T_1 T_2$ on $\mathcal{H}^{\otimes 3}$, and $\|T\| \leq 1$, then by [1] $P_n \geq 0, n \geq 2$. Moreover in this case for any $n \geq 2$

$$\ker P_n = \sum_{i=1}^{n-1} \ker(\mathbf{1}_{\mathcal{H}^{\otimes n}} + T_i)$$

and the kernel of the Fock representation is generated as a two-sided $*$ -ideal by $\ker(\mathbf{1}_{\mathcal{H}^{\otimes 2}} + T)$, see [2]. Furthermore, if T is braided and $\ker(\mathbf{1}_{\mathcal{H}^{\otimes 2}} + T) \neq \{0\}$, the two-sided ideal $\mathcal{I}_2 \subset \mathcal{T}(\mathcal{H})$ generated by $\ker(\mathbf{1}_{\mathcal{H}^{\otimes 2}} + T)$ is invariant with respect to multiplication by any $a_i^*, i = 1, \dots, d$. i.e.

$$e_i^* \otimes \mathcal{I}_2 \subset \mathcal{I}_2 + \mathcal{I}_2 \otimes \mathcal{H}^* \quad (5)$$

Ideals $I \subset \mathcal{T}(\mathcal{H})$ satisfying (5) are called *Wick ideals*, see [3]. It was shown that homogeneous Wick ideals, i.e. those ones which are generated by subspaces in $\mathcal{H}^{\otimes n}$, are annihilated by the Fock representation, see [3]. In [2] the authors prove that if the operator T is braided then existence of homogeneous Wick ideals is necessary for existence of Wick ideals in general. If T is a braided contraction, then any homogeneous Wick ideal of higher degree is contained in a largest quadratic one, see [2]. Note that for some Wick algebras (e.g. Wick algebras associated with twisted canonical commutation relations of W. Pusz and S.L. Woronowicz, see [3, 7]; quonic commutation relations, see [4] and others) their quadratic Wick ideals are contained in their $*$ -radicals, i.e. such ideals are annihilated by any bounded $*$ -representation of the corresponding algebra.

In this paper we investigate the structure of homogeneous Wick ideals of higher degrees. We present a method how to construct a homogeneous Wick ideal \mathcal{I}_{n+1} of degree $n+1$ out of a homogeneous Wick ideal \mathcal{I}_n of degree n so that $\mathcal{I}_{n+1} \subset \mathcal{I}_n$. We show that in some particular cases our procedure allows to get a description of largest homogeneous Wick ideals of higher degrees having generators of the largest quadratic Wick ideal only. Finally we study classes of $*$ -representations of Wick version of CCR annihilating certain homogeneous Wick ideals of degree higher than 2.

2 Wick ideals: basic definitions and properties.

The notion of Wick ideal in quadratic Wick algebra was presented in [3]. It was proposed as a natural way to introduce additional relations between generators $a_i, i = 1, \dots, d$, which are consistent with the basic relations of the algebra.

Following [3] we will work with the canonical realisation of $W(T)$ as a quotient of the tensor algebra $\mathcal{T}(\mathcal{H}, \mathcal{H}^*)$ given by (2). In this realisation the subalgebra generated by $a_i, i = 1, \dots, d$, is identified with $\mathcal{T}(\mathcal{H})$.

Definition 1. A two-sided ideal $\mathcal{I} \subset \mathcal{T}(\mathcal{H})$ is called a *Wick ideal* if

$$\mathcal{T}(\mathcal{H}^*) \otimes \mathcal{I} \subset \mathcal{I} \otimes \mathcal{T}(\mathcal{H}^*).$$

If the Wick ideal \mathcal{I} is generated by a subspace $\mathcal{I}_0 \subset \mathcal{H}^{\otimes n}$, then \mathcal{I} is called a homogeneous Wick ideal of degree n .

It is easy to verify the following criteria for a two-sided ideal \mathcal{I} to be a Wick one, see [3].

Proposition 1. *A two-sided ideal $\mathcal{I} \subset \mathcal{T}(\mathcal{H})$ is Wick iff*

$$\mathcal{H}^* \otimes \mathcal{I} \subset \mathcal{I} + \mathcal{I} \otimes \mathcal{H}^*.$$

Remark 2. If an ideal $\mathcal{I} \subset \mathcal{T}(\mathcal{H})$ is generated by a subspace $\mathcal{I}_0 \subset \mathcal{H}^{\otimes n}$, then it is Wick iff

$$\mathcal{H}^* \otimes \mathcal{I}_0 \subset \mathcal{I}_0 + \mathcal{I}_0 \otimes \mathcal{H}^*$$

It is important from the representation theory point of view to get a precise description of generators of homogeneous Wick ideals of degrees higher than 2. The first step in this direction was done in [6]. Namely, in this paper the following statement was proved.

Proposition 2. *Let T be a braided contraction and let $\mathcal{I}_2 \subset \mathcal{H}^{\otimes 2}$ generate the largest quadratic Wick ideal. Then*

$$\mathcal{I}_3 = (\mathbf{1}_{\mathcal{H}^{\otimes 3}} - T_1 T_2)(\mathcal{I}_2 \otimes \mathcal{H})$$

generates the largest Wick ideal of degree 3.

Below we will often say "homogeneous Wick ideal of degree n " meaning a linear subspace in $\mathcal{H}^{\otimes n}$ generating this ideal.

3 Homogeneous Wick ideals

We start with a simple observation, showing that the product of homogeneous Wick ideals is again a homogeneous Wick ideal.

Proposition 3. *Let \mathcal{J}_n and \mathcal{J}_k be homogeneous Wick ideals of degree n and k respectively, then their tensor product $\mathcal{J}_n \otimes \mathcal{J}_k$ is a homogeneous Wick ideal of degree $n + k$.*

Proof. Indeed, since for a Wick ideal one has

$$\mathcal{H}^* \otimes \mathcal{I} \subset \mathcal{I} + \mathcal{I} \otimes \mathcal{H}^*$$

we get

$$\begin{aligned} \mathcal{H}^* \otimes (\mathcal{J}_n \otimes \mathcal{J}_k) &\subset (\mathcal{J}_n + \mathcal{J}_n \otimes \mathcal{H}^*) \otimes \mathcal{J}_k = \mathcal{J}_n \otimes \mathcal{J}_k + \mathcal{J}_n \otimes \mathcal{H}^* \otimes \mathcal{J}_k \subset \\ &\subset \mathcal{J}_n \otimes \mathcal{J}_k + \mathcal{J}_n \otimes \mathcal{J}_k + \mathcal{J}_n \otimes \mathcal{J}_k \otimes \mathcal{H}^* = \mathcal{J}_n \otimes \mathcal{J}_k + \mathcal{J}_n \otimes \mathcal{J}_k \otimes \mathcal{H}^*. \end{aligned}$$

Thus, $\mathcal{J}_n \otimes \mathcal{J}_k \subset \mathcal{H}^{\otimes(n+k)}$ is a Wick ideal. \square

The following proposition was proved in [3] for quadratic Wick ideals and in [6] in general case.

Proposition 4. *Let $P: \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$ be a projection. The subspace $\mathcal{I} = P(\mathcal{H}^{\otimes n})$ generates a Wick ideal iff*

1. $R_n P = 0$ (equality in $\mathcal{H}^{\otimes n}$),
2. $[\mathbf{1}_{\mathcal{H}} \otimes (\mathbf{1}_{\mathcal{H}^{\otimes n}} - P)]T_1 T_2 \cdots T_n [P \otimes \mathbf{1}_{\mathcal{H}}] = 0$ (equality in $\mathcal{H}^{\otimes n+1}$).

Moreover, if T is braided and P is the projection onto $\ker R_n$, the second condition holds automatically and hence $\ker R_n$ generates the largest homogenous Wick ideal of degree n .

Remark 3. Note, that the second condition of Proposition 4 means

$$T_1 T_2 \cdots T_n (\mathcal{I} \otimes \mathcal{H}) \subset \mathcal{H} \otimes \mathcal{I}.$$

Lemma 1. *Let $\mathcal{I} \subset \mathcal{H}^{\otimes n}$ generate a homogeneous Wick ideal, then*

$$(\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1 T_2 \cdots T_n)(\mathcal{I} \otimes \mathcal{H}) \subset \ker R_{n+1}.$$

Proof. Let $X \in \mathcal{I}$. Then $X \in \ker R_n$. Note that

$$R_{n+1} = R_n \otimes \mathbf{1}_{\mathcal{H}} + T_1 T_2 \cdots T_n = \mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} + T_1 (\mathbf{1}_{\mathcal{H}} \otimes R_n)$$

Then for any $i = 1, \dots, d$ one has

$$\begin{aligned} R_{n+1}(\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1 T_2 \cdots T_n)(X \otimes e_i) &= \\ &= R_{n+1}(X \otimes e_i) - R_{n+1} T_1 T_2 \cdots T_n (X \otimes e_i) = \\ &= (R_n \otimes \mathbf{1}_{\mathcal{H}} + T_1 T_2 \cdots T_n)(X \otimes e_i) \\ &\quad - (\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} + T_1 (\mathbf{1}_{\mathcal{H}} \otimes R_n)) T_1 T_2 \cdots T_n (X \otimes e_i) = \\ &= T_1 T_2 \cdots T_n (X \otimes e_i) - T_1 T_2 \cdots T_n (X \otimes e_i) \\ &\quad - T_1 (\mathbf{1}_{\mathcal{H}} \otimes R_n) T_1 T_2 \cdots T_n (X \otimes e_i) = 0, \end{aligned}$$

where we used

$$T_1 T_2 \cdots T_n (\mathcal{I} \otimes \mathcal{H}) \subset \mathcal{H} \otimes \mathcal{I} \subset \mathcal{H} \otimes \ker R_n = \ker (\mathbf{1}_{\mathcal{H}} \otimes R_n).$$

□

The following corollary is immediate.

Corollary 1. *If the operator T is braided, then*

$$(\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1 T_2 \cdots T_n)(\ker R_n \otimes \mathcal{H}) \subset \ker R_{n+1}.$$

Below we will use the following simple observation

Lemma 2. *Let T be braided. Then for any $n \geq 2$ and $k \leq n-1$*

$$(T_1 T_2 \cdots T_n)(T_1 T_2 \cdots T_k) = (T_2 T_3 \cdots T_{k+1})(T_1 T_2 \cdots T_n).$$

Proof. Evidently it is enough to check that

$$T_1 T_2 \cdots T_n T_j = T_{j+1} T_1 T_2 \cdots T_n, \quad 1 \leq j \leq n-1.$$

Indeed, since $T_i T_j = T_j T_i$ when $|i-j| \geq 2$ and $T_j T_{j+1} T_j = T_{j+1} T_j T_{j+1}$ we get

$$\begin{aligned} T_1 T_2 \cdots T_n T_j &= T_1 T_2 \cdots T_{j-1} T_j T_{j+1} T_j T_{j+2} \cdots T_n = \\ &= T_1 T_2 \cdots T_{j-1} T_{j+1} T_j T_{j+1} T_{j+2} \cdots T_n = \\ &= T_{j+1} T_1 T_2 \cdots T_n. \end{aligned}$$

□

The following proposition gives a procedure to compute generators of certain homogeneous Wick ideals of degree $n+1$ out of generators of Wick ideals of degree n when T is braided.

Proposition 5. *Let T be braided and $\mathcal{I}_n \subset \mathcal{H}^{\otimes n}$ generate a homogeneous Wick ideal of degree n . Then*

$$\mathcal{I}_{n+1} = (\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1 T_2 \cdots T_n)(\mathcal{I}_n \otimes \mathcal{H})$$

generates a homogeneous Wick ideal of degree $n+1$.

Proof. According to Lemma 1

$$(\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1 T_2 \cdots T_n)(\mathcal{I}_n \otimes \mathcal{H}) \subset \ker R_{n+1}$$

so, it remains to prove that

$$T_1 T_2 \cdots T_{n+1}(\mathcal{I}_{n+1} \otimes \mathcal{H}) \subset \mathcal{H} \otimes \mathcal{I}_{n+1}. \quad (6)$$

Indeed

$$\begin{aligned} T_1 T_2 \cdots T_{n+1}(\mathcal{I}_{n+1} \otimes \mathcal{H}) &= T_1 T_2 \cdots T_{n+1}(\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1 T_2 \cdots T_n)(\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) = \\ &= (T_1 T_2 \cdots T_{n+1} - T_1 T_2 \cdots T_{n+1} T_1 T_2 \cdots T_n)(\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) = \\ &= (T_1 T_2 \cdots T_{n+1} - T_2 T_3 \cdots T_{n+1} T_1 T_2 \cdots T_{n+1})(\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) = \\ &= (\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_2 T_3 \cdots T_{n+1}) T_1 T_2 \cdots T_{n+1}(\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) = \\ &= (\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_2 T_3 \cdots T_{n+1}) T_1 T_2 \cdots T_n(\mathcal{I}_n \otimes T(\mathcal{H} \otimes \mathcal{H})) \subset \\ &\subset (\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_2 T_3 \cdots T_{n+1}) T_1 T_2 \cdots T_n(\mathcal{I}_n \otimes \mathcal{H} \otimes \mathcal{H}) \subset \\ &\subset (\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_2 T_3 \cdots T_{n+1})(\mathcal{H} \otimes \mathcal{I}_n \otimes \mathcal{H}) = \\ &= \mathcal{H} \otimes (\mathbf{1}_{\mathcal{H}^{\otimes n}} - T_1 T_2 \cdots T_n)(\mathcal{I}_n \otimes \mathcal{H}) = \\ &= \mathcal{H} \otimes \mathcal{I}_{n+1}. \end{aligned}$$

□

Next aim is to describe largest Wick ideals.

Lemma 3. *Let T satisfy the braid relation. Then*

$$R_{n+1}T_1T_2\cdots T_n = T_1T_2\cdots T_n + T_1^2T_2\cdots T_n(R_n \otimes \mathbf{1}_{\mathcal{H}}) \quad (7)$$

Proof. Indeed

$$\begin{aligned} R_{n+1}T_1T_2\cdots T_n &= \\ &= T_1T_2\cdots T_n + T_1(\mathbf{1}_{\mathcal{H}^{\otimes n+1}} + T_2 + T_2T_3 + \cdots + T_2T_3\cdots T_n)T_1T_2\cdots T_n = \\ &= T_1T_2\cdots T_n + T_1^2T_2\cdots T_n(\mathbf{1}_{\mathcal{H}^{\otimes n+1}} + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1}) = \\ &= T_1T_2\cdots T_n + T_1^2T_2\cdots T_n(R_n \otimes \mathbf{1}_{\mathcal{H}}). \end{aligned}$$

□

Lemma 4. *Let T be braided. Then*

$$R_{n+1}(\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1T_2\cdots T_n) = (\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1^2T_2\cdots T_n)(R_n \otimes \mathbf{1}_{\mathcal{H}}).$$

Proof. By the previous Lemma

$$\begin{aligned} R_{n+1} - R_{n+1}T_1T_2\cdots T_n &= \\ &= R_n \otimes \mathbf{1}_{\mathcal{H}} + T_1T_2\cdots T_n - T_1T_2\cdots T_n - T_1^2T_2\cdots T_n(R_n \otimes \mathbf{1}_{\mathcal{H}}) = \\ &= (1 - T_1^2T_2\cdots T_n)(R_n \otimes \mathbf{1}_{\mathcal{H}}). \end{aligned}$$

□

Let $\mathcal{K}_2 = \ker R_2$ and

$$\mathcal{K}_{m+1} = (\mathbf{1}_{\mathcal{H}^{\otimes(m+1)}} - T_1T_2\cdots T_m)(\mathcal{K}_m \otimes \mathcal{H}), \quad m \geq 2.$$

Since by (6)

$$\mathcal{K}_{m+1} \subset \mathcal{H} \otimes \mathcal{K}_m + \mathcal{K}_m \otimes \mathcal{H},$$

the Wick ideals generated by \mathcal{K}_m , $m \geq 2$, form a nested sequence

$$\langle \mathcal{K}_2 \rangle \supset \langle \mathcal{K}_3 \rangle \supset \cdots \supset \langle \mathcal{K}_m \rangle \supset \cdots$$

Proposition 6. *Suppose that T is braided and for any $m \geq 2$*

$$\ker(\mathbf{1}_{\mathcal{H}^{\otimes(m+1)}} - T_1T_2\cdots T_m) = \{0\} \quad \text{and} \quad \ker(\mathbf{1}_{\mathcal{H}^{\otimes(m+1)}} - T_1^2T_2\cdots T_m) = \{0\}.$$

Then

$$\mathcal{K}_m = \ker R_m, \quad m \geq 2,$$

and hence \mathcal{K}_m generates the largest homogeneous Wick ideals of degree m for any $m \geq 2$.

Proof. Suppose that $\dim \mathcal{H} = d$. If $\mathbf{1}_{\mathcal{H}^{\otimes m+1}} - T_1 T_2 \cdots T_m$, $m \geq 2$ are invertible, by the definition of \mathcal{K}_m we have

$$\dim \mathcal{K}_m = d \cdot \dim \mathcal{K}_{m-1} = d^{m-2} \cdot \dim \ker R_2.$$

As $\mathcal{K}_m \subset \ker R_m$ (by Lemma 1) it remains to see that for any $m \geq 2$ one has

$$\dim \ker R_m = d \cdot \dim \ker R_{m-1} = \dots = d^{m-2} \dim \ker R_2$$

But this immediatly follows from the equality

$$R_{m+1}(\mathbf{1}_{\mathcal{H}^{\otimes m+1}} - T_1 T_2 \cdots T_m) = (\mathbf{1}_{\mathcal{H}^{\otimes m+1}} - T_1^2 T_2 \cdots T_m)(R_m \otimes \mathbf{1}_{\mathcal{H}})$$

and invertibility of the operators $\mathbf{1}_{\mathcal{H}^{\otimes m+1}} - T_1 T_2 \cdots T_m$ and $\mathbf{1}_{\mathcal{H}^{\otimes m+1}} - T_1^2 T_2 \cdots T_m$.

Hence, $\dim \ker R_m = \dim \ker \mathcal{K}_m$ and

$$\mathcal{K}_m = \ker R_m, \quad m \geq 2.$$

□

Lemma 5. *Let T be braided and $\|T_1 T_2 T_1\| = q < 1$, $\|T\| = 1$. Then $\ker R_m = \mathcal{K}_m$ for any $m \geq 2$.*

Proof. By Propostion 6 it is enough to see that

$$1 \notin \sigma(T_1 T_2 \cdots T_n) \quad \text{and} \quad 1 \notin \sigma(T_1^2 T_2 \cdots T_n).$$

Indeed, since $T_i T_j = T_j T_i$, $|i - j| \geq 2$, and $\|T_i\| = 1$, $i = 1, \dots, n$, we get

$$(T_1 T_2 T_3 \cdots T_n)^2 = (T_1 T_2 T_1)(T_3 T_4 \cdots T_n T_2 T_3 \cdots T_n)$$

implying

$$\|(T_1 T_2 \cdots T_n)^2\| \leq q < 1$$

and hence $1 \notin \sigma(T_1 T_2 \cdots T_n)$.

Analogously,

$$(T_1^2 T_2 \cdots T_n)^2 = T_1(T_1 T_2 T_1)(T_3 T_4 \cdots T_n T_1 T_2 \cdots T_n)$$

and $\|(T_1^2 T_2 \cdots T_n)^2\| \leq q < 1$ giving $1 \notin \sigma(T_1^2 T_2 \cdots T_n)$.

□

In what follows we shall often say ideal \mathcal{K}_m meaning the ideal generated by \mathcal{K}_m .

In general, see Section 3 and Section 4, largest homogeneous Wick ideals do not coincide with the ideals \mathcal{K}_m . However a direct calculations in MATHEMATICA shows that for some Wick algebras, including Wick versions of CCR, twisted CCR, twisted CAR and quonic commutation relations, see [3], the following conjecture is true.

Conjecture 1. If T is braided then

$$\ker R_{n+1} = (\mathbf{1}_{\mathcal{H}^{\otimes(n+1)}} - T_1 T_2 \cdots T_n)(\ker R_n \otimes \mathcal{H}) + \ker R_{n-2} \otimes \ker R_2.$$

4 Homogeneous ideals of Wick version of quon commutation relations

Here we apply results of the previous section to get a description of homogeneous ideals the Wick algebra, \mathcal{A}_2^q , associated with quon commutation relations with two degrees of freedom, see [4]. Recall that \mathcal{A}_2^q is a $*$ -algebra generated by elements $a_i, a_i^*, i = 1, 2$, satisfying commutation relations of the form

$$\begin{aligned} a_i^* a_i &= 1 + q a_i a_i^*, \quad i = 1, 2, \\ a_1^* a_2 &= \lambda a_2 a_1^*, \end{aligned}$$

where q, λ are parameters such that $0 < q < 1, |\lambda| = 1$. In this case $\dim \mathcal{H} = 2$ and the operator T is given by

$$\begin{aligned} T e_i \otimes e_i &= q e_i \otimes e_i, \quad i = 1, 2, \\ T e_1 \otimes e_2 &= \bar{\lambda} e_2 \otimes e_1, \quad T e_2 \otimes e_1 = \lambda e_1 \otimes e_2 \end{aligned} \quad (8)$$

It is easy to verify that T is braided, $\|T\| = 1$ for any $q \in (0, 1), \lambda \in \mathbb{C}, |\lambda| = 1$ and

$$\ker(\mathbf{1}_{\mathcal{H}^{\otimes 2}} + T) = \mathbb{C} \langle A = e_2 \otimes e_1 - \lambda e_1 \otimes e_2 \rangle.$$

Proposition 7. *Let $T: \mathcal{H}^{\otimes 2} \rightarrow \mathcal{H}^{\otimes 2}$ be defined by (8) and $\dim \mathcal{H} = 2$. Then for any $m \geq 2$, $\ker R_m = \mathcal{K}_m$ is the largest homogeneous Wick ideal of degree m .*

Proof. By Lemma 5 it is enough to show that $\|T_1 T_2 T_1\| < 1$. Indeed, it is easy to see that for the standard orthonormal basis of $\mathcal{H}^{\otimes 3}$ one has

$$\begin{aligned} T_1 T_2 T_1 e_i \otimes e_i \otimes e_i &= q^3 e_i \otimes e_i \otimes e_i, \quad i = 1, 2 \\ T_1 T_2 T_1 e_1 \otimes e_1 \otimes e_2 &= q \bar{\lambda}^2 e_2 \otimes e_1 \otimes e_1, \quad T_1 T_2 T_1 e_2 \otimes e_1 \otimes e_1 = q \lambda^2 e_1 \otimes e_1 \otimes e_2 \\ T_1 T_2 T_1 e_2 \otimes e_2 \otimes e_1 &= q \lambda^2 e_1 \otimes e_2 \otimes e_2, \quad T_1 T_2 T_1 e_1 \otimes e_2 \otimes e_2 = q \bar{\lambda}^2 e_2 \otimes e_2 \otimes e_1 \\ T_1 T_2 T_1 e_1 \otimes e_2 \otimes e_1 &= q e_1 \otimes e_2 \otimes e_1, \quad T_1 T_2 T_1 e_2 \otimes e_1 \otimes e_2 = q e_2 \otimes e_1 \otimes e_2. \end{aligned}$$

Hence $\|T_1 T_2 T_1\| = q < 1$. □

Remark 4. 1. For Wick quonic relations with three generators Lemma 5 cannot be applied, since in this case $\|T_1 T_2 T_1\| = 1$. However, since T is a braided contraction we have by Proposition 2

$$\ker R_3 = (\mathbf{1}_{\mathcal{H}^{\otimes 3}} - T_1 T_2)(\ker R_2 \otimes \mathcal{H})$$

and one can apply Proposition 6 to show that in this case $\mathcal{K}_m = \ker R_m$, $m \geq 2$ as well.

2. Computations in MATHEMATICA show that for Wick quonic relations with four or more generators the ideals \mathcal{K}_m do not coincide with $\ker R_m$ for $m > 3$.

4.1 *-Representations of \mathcal{A}_2^q , annihilating homogeneous ideals

In this section we show that any *-representation of the Wick quonic relations annihilating \mathcal{K}_m for some fixed $m \geq 2$ annihilates the ideal \mathcal{K}_2 .

First we recall that for any bounded *-representation π of \mathcal{A}_2^q one has $\pi(\mathcal{K}_2) = 0$, see [5]. Indeed, it is easy to verify, that if $A = a_2 a_1 - \lambda a_1 a_2$, then

$$a_1^* A = \lambda q A a_1^*, \quad a_2^* A = \bar{\lambda} q A a_2^*$$

implying that $A^* A = q^2 A A^*$. Evidently, the only bounded operator A satisfying such relation is the zero one.

Proposition 8. *Let π be an irreducible *-representation (possibly unbounded) of \mathcal{A}_2^q such that $\pi(\mathcal{K}_m) = \{0\}$ for some $m \geq 3$. Then $\pi(A) = 0$ and hence $\pi(\mathcal{K}_2) = 0$.*

Proof. By Propositions 7 for any $m \geq 3$ the ideal \mathcal{K}_m coincides with the largest homogeneous ideal of degree m .

Let $m = 2k$, for some $k > 1$. Then, since the product of homogeneous Wick ideals is a homogeneous Wick ideal, we get

$$(\ker R_2)^{\otimes k} \subset \ker R_{2k} = \mathcal{K}_m.$$

So if $\pi(\mathcal{K}_m) = \{0\}$, then $\pi(A^k) = 0$ and hence $\ker \pi(A) \neq \{0\}$. Further, $A^* A = q^2 A A^*$ implies that $\ker \pi(A) = \ker \pi(A^*)$ and from

$$A a_1^* = \bar{\lambda} q^{-1} a_1^* A, \quad A a_2^* = \lambda q^{-1} a_2^* A, \quad A^* a_1 = \bar{\lambda} q a_1 A^*, \quad A^* a_2 = \lambda q a_2 A^*$$

we obtain that $\ker \pi(A) = \ker \pi(A^*)$ is invariant with respect to $\pi(a_i)$ and $\pi(a_i^*)$, $i = 1, 2$. Thus if π is irreducible, $\pi(A) = \{0\}$.

Suppose now that $\pi(\mathcal{K}_m) = \{0\}$ and $m = 2k + 1$, for fixed $k \geq 1$. Then as above $A^{k+1} \in \mathcal{K}_{m+1}$. Since $\langle \mathcal{K}_{m+1} \rangle \subset \langle \mathcal{K}_m \rangle$ we get $\pi(A^{k+1}) = 0$ and repeating the arguments from the previous paragraph we obtain $\pi(A) = 0$. \square

We refer the reader to [8] for definitions and facts about unbounded *-representations of *-algebras. Note that such representations can be rather complicated and one usually restricts oneself to a subclass of "well-behaved" representations. For Lie algebras natural well-behaved representations are integrable representations i.e. those which can be integrated to a unitary representation of the corresponding Lie group (see for example [8, Section 10]).

5 *-Representations of Wick version of CCR annihilating homogenous ideals

In this Section we consider a Wick version of CCR, denoted below by \mathcal{A}_d^0 , and given by

$$\mathcal{A}_d^0 = \mathbb{C} \langle a_i, a_i^* \mid a_i^* a_j = \delta_{ij} \mathbf{1} + a_j a_i^*, \quad i, j = 1, \dots, d \rangle.$$

In this case T is the flip operator

$$Te_i \otimes e_j = e_j \otimes e_i, \quad i, j = 1, \dots, d$$

and the largest quadratic ideal $\mathcal{K}_2 = \ker R_2$ is generated by the elements

$$A_{ij} = e_j \otimes e_i - e_i \otimes e_j, \quad i \neq j, \quad i, j = 1, \dots, d.$$

The action of the operator $T_1 T_2 \cdots T_k$ on a product of the form $B \otimes e_i$, $B \in \mathcal{H}^k$, $i = 1, \dots, d$, is the following

$$(T_1 T_2 \cdots T_k)(B \otimes e_i) = e_i \otimes B, \quad i = 1, \dots, d.$$

Thus if the homogeneous Wick ideal \mathcal{K}_m is generated by a family $\{B_j, j \in \mathcal{J}\}$, then

$$\mathcal{K}_{m+1} = \langle e_i \otimes B_j - B_j \otimes e_i, \quad i = 1, \dots, d, \quad j \in \mathcal{J} \rangle$$

Recall that

$$e_i^* \otimes B_j = \mu_0(e_i^*)(R_m B_j + \sum_{k=1}^d T_1 T_2 \cdots T_m(B_j \otimes e_k) \otimes e_k^*), \quad i = 1, \dots, d, \quad j \in \mathcal{J}.$$

Since

$$T_1 T_2 \cdots T_m(B_j \otimes e_k) = e_k \otimes B_j, \quad R_n B_j = 0,$$

and $\mu_0(e_i^*)e_k \otimes X = \delta_{ik}X$ for any $X \in \mathcal{T}(\mathcal{H})$, we get

$$e_i^* \otimes B_j = B_j \otimes e_i^*, \quad i = 1, \dots, d, \quad j \in \mathcal{J}.$$

In other words if we consider the quotient of \mathcal{A}_d^0 by the homogeneous Wick ideal \mathcal{K}_{m+1} we obtain the following commutation relations between generators of the algebra and generators of the ideal \mathcal{K}_m

$$a_i^* B_j = B_j a_i^*, \quad a_i B_j = B_j a_i, \quad i = 1, \dots, d, \quad j \in \mathcal{J}.$$

We intend to study representations of \mathcal{A}_2^0 annihilating the ideals \mathcal{K}_m , $m = 2, 3, 4$.

5.1 Representations of \mathcal{A}_2^0 annihilating quadratic and cubic ideals

Below we assume $d = 2$. The quadratic ideal \mathcal{K}_2 is generated by $a_1 \otimes a_2 - a_2 \otimes a_1$ and the quotient $\mathcal{A}_2^0/\mathcal{K}_2$ is the Weyl algebra with two degrees of freedom. Note that it is a quotient of the universal enveloping of the Heisenberg algebra. The unique irreducible well-behaved representation of the Weyl algebra (by well-behaved we mean a representation which can be integrated to a unitary representation of the Heisenberg Lie group), is the Fock representation: the space of the representation is $\mathcal{H} = l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}_+)$ and

$$a_1 = a \otimes \mathbf{1}, \quad a_2 = \mathbf{1} \otimes a,$$

where $ae_n = \sqrt{n+1}e_{n+1}$, $n \in \mathbb{Z}_+$, and $\{e_n, n \in \mathbb{Z}_+\}$ is the standard orthonormal basis in $l_2(\mathbb{Z}_+)$.

Now we study irreducible representations of \mathcal{A}_2^0 which annihilate the ideal \mathcal{K}_3 . The ideal \mathcal{K}_3 is generated by the elements

$$Aa_1 - a_1A, \quad Aa_2 - a_2A$$

with $A = a_2a_1 - a_1a_2$.

Since $a_i^*A = Aa_i^*$, $i = 1, 2$, we conclude that A belongs to the center of the quotient $\mathcal{A}_2^0/\mathcal{K}_3$.

For a well-behaved irreducible representation π , we assume that A commutes with a_i, a_i^* strongly (i.e. A is closable on the domain of the representation and if $A = U|A|$ is the polar decomposition of A then U and all spectral projections of $|A|$ belongs to the strong commutant of the family $\{a_1, a_1^*, a_2, a_2^*\}$, [8]) and by the Schur lemma we have $A = x\mathbf{1}$, $x \in \mathbb{C}$ (we denote the operators of the representation by the same letters as the corresponding elements of the algebra). Thus, the problem of classification of such irreducible representations is reduced to the classification of irreducible representations of the following family of commutation relations

$$\begin{aligned} a_i^*a_i - a_ia_i^* &= \mathbf{1}, \quad i = 1, 2, \\ a_1^*a_2 &= a_2a_1^* \quad a_2a_1 - a_1a_2 = x\mathbf{1}. \end{aligned} \quad (9)$$

Denote by $A_{2,x}$ the $*$ -algebra generated by relations (9) and by $A_{2,0}$ the $*$ -algebra generated by CCR with two degrees of freedom.

Proposition 9. *The $*$ -algebras $A_{2,x}$ and $A_{2,0}$ are isomorphic for any $x \in \mathbb{C}$.*

Proof. For any fixed $x \in \mathbb{C}$ let

$$d_1 = a_1 \quad \text{and} \quad d_2 = \left(1 + |x|^2\right)^{-\frac{1}{2}} a_2 - xa_1^*.$$

Then it is easy to verify that d_1, d_2 generate $A_{2,x}$ and

$$d_i^*d_i - d_id_i^* = \mathbf{1}, \quad i = 1, 2, \quad d_1^*d_2 = d_2d_1^*, \quad d_2d_1 = d_1d_2. \quad (10)$$

Conversely, let c_1, c_2 be generators of $A_{2,0}$ satisfying (10). Put

$$b_1 = c_1, \quad b_2 = \left(1 + |x|^2\right)^{\frac{1}{2}} c_2 + xc_1^*$$

Then b_1, b_2 satisfy (9) and generate $A_{2,0}$. Hence $A_{2,x} \simeq A_{2,0}$. \square

It follows from the uniqueness of irreducible well-behaved representation of CCR with two degrees of freedom that there exists a unique, up to a unitary equivalence, irreducible representation of (10) defined on $l_2(\mathbb{Z}_+)^{\otimes 2}$ by the formulas

$$d_1 = a \otimes \mathbf{1}, \quad d_2 = \mathbf{1} \otimes a.$$

Below by *well-behaved* representation of $A_{2,x}$ we mean a well-behaved representation of $A_{2,0} \simeq A_{2,x}$. Applying Proposition 9 we get the following result.

Theorem 1. *For any $x \in \mathbb{C}$ there exists a unique, up to unitary equivalence, irreducible well-behaved representation of $A_{2,x}$ given by*

$$\begin{aligned} a_1 &= a \otimes \mathbf{1}, \\ a_2 &= \sqrt{1 + |x|^2} \mathbf{1} \otimes a + xa^* \otimes \mathbf{1}. \end{aligned}$$

Evidently in the case $x = 0$ we get the Fock representation, annihilating \mathcal{K}_2 .

5.2 Representations annihilating \mathcal{K}_4

Let us describe representations of \mathcal{A}_2^0 which annihilate the ideal \mathcal{K}_4 . Recall that

$$\mathcal{K}_4 = \langle B_i a_j - a_j B_i, \quad i, j = 1, 2 \rangle,$$

where $B_i = Aa_i - a_i A$, $i = 1, 2$, are generators of \mathcal{K}_3 . Since

$$a_j^* B_i = B_i a_j^*, \quad i = 1, 2,$$

the elements B_1, B_2 belong to the center of the quotient $\mathcal{A}_2^0/\mathcal{K}_4$. Identifying again the elements with their images in a representation π annihilating \mathcal{K}_4 we require that for a well-behaved irreducible representation

$$B_1 = Aa_1 - a_1 A = x_1 \mathbf{1}, \quad B_2 = Aa_2 - a_2 A = x_2 \mathbf{1}$$

for some $x_1, x_2 \in \mathbb{C}$. Note also that in \mathcal{A}_2^0 we have $a_i^* A = Aa_i^*$, $i = 1, 2$.

5.2.1 Representations with $x_1 \neq 0$.

Fix $(x_1, x_2) \in \mathbb{C}^2$ with $x_1 \neq 0$ and consider the $*$ -algebra A_{x_1, x_2} , generated by elements a_1, a_2, A satisfying the following commutation relations

$$\begin{aligned} a_i^* a_i - a_i a_i^* &= 1, \\ a_1^* a_2 &= a_2 a_1^*, \quad A = a_2 a_1 - a_1 a_2, \\ Aa_i - a_i A &= x_i \mathbf{1}, \quad a_i^* A = Aa_i^*, \quad i = 1, 2. \end{aligned} \tag{11}$$

Let

$$\begin{aligned} d_1 &= a_1 \\ d_2 &= |x_1|^{-1} (A - x_1 a_1^*) \\ d_3 &= \left(1 + \frac{|x_2|^2}{|x_1|^2}\right)^{-\frac{1}{2}} \left(a_2 + \frac{x_2}{|x_1|} d_2^* - \frac{\bar{x}_1}{2} d_2^2 - |x_1| d_1^* d_2 - \frac{x_1}{2} (d_1^*)^2\right) \end{aligned} \tag{12}$$

Below we show that the elements d_i , $i = 1, 2, 3$, generate A_{x_1, x_2} and satisfy CCR with three degrees of freedom.

First we establish some commutation relations between a_i and d_j , $i, j = 1, 2$.

Lemma 6. *The elements a_1, a_2, d_1, d_2 satisfy the following relations*

$$\begin{aligned} d_1^* a_2 &= a_2 d_1^*, \\ a_2 d_1 - d_1 a_2 &= |x_1| d_2 + x_1 d_1^*, \\ a_2^* d_2 &= d_2 a_2^* + x_1 d_2^* + |x_1| d_1, \\ a_2 d_2 &= d_2 a_2 - \frac{x_2}{|x_1|}. \end{aligned} \tag{13}$$

Proof. The first two relations follow directly from the definition of d_1, d_2 and (11). Further

$$\begin{aligned} |x_1| a_2 d_2 &= a_2 A - x_1 a_2 a_1^* = A a_2 - x_2 - x_1 a_1^* a_2 = \\ &= (A a_2 - x_1 a_1^*) a_2 - x_2 = |x_1| d_2 a_2 - x_2, \end{aligned}$$

and

$$\begin{aligned} |x_1| a_2^* d_2 &= a_2^* A - x_1 a_2^* a_1^* = A a_2^* - x_1 (a_1^* a_2^* - A^*) = \\ &= (A - x_1 a_1^*) a_2^* + x_1 A^* = |x_1| d_2 a_2^* + x_1 (|x_1| d_2^* + \bar{x}_1 d_1) = \\ &= |x_1| (d_2 a_2^* + x_1 d_2^* + |x_1| d_1). \end{aligned}$$

□

Lemma 7. *The elements $d_i, d_i^*, i = 1, 2, 3$, generate A_{x_1, x_2} and satisfy CCR with three degrees of freedom, i.e. for any $i = 1, 2, 3$ and $i \neq j$*

$$d_i^* d_i - d_i d_i^* = 1, \quad d_i^* d_j = d_j d_i^*, \quad d_i d_j = d_j d_i. \tag{14}$$

Proof. It easily follows from (12) that

$$\begin{aligned} a_1 &= d_1, \\ A &= |x_1| d_2 + x_1 d_1^*, \\ a_2 &= \left(1 + \frac{|x_2|^2}{|x_1|^2}\right)^{\frac{1}{2}} d_3 - \frac{x_2}{|x_1|} d_2^* + \frac{\bar{x}_1}{2} d_2^2 + |x_1| d_1^* d_2 + \frac{x_1}{2} (d_1^*)^2 \end{aligned} \tag{15}$$

proving that A_{x_1, x_2} is generated by d_1, d_2, d_3 .

Further

$$\begin{aligned} |x_1| d_2 d_1 &= (A - x_1 a_1^*) a_1 = A a_1 - x_1 (1 + a_1 a_1^*) = \\ &= a_1 A + x_1 - x_1 - x_1 a_1 a_1^* = a_1 (A - x_1 a_1^*) = |x_1| d_1 d_2 \end{aligned}$$

$$|x_1| d_1^* d_2 = a_1^* (A - x_1 a_1^*) = A a_1^* - x_1 (a_1^*)^2 = (A - x_1 a_1^*) a_1^* = |x_1| d_2 d_1^*$$

Now let us check that $d_2^* d_2 - d_2 d_2^* = 1$

$$\begin{aligned} |x_1|^2 d_2^* d_2 &= (A^* - \bar{x}_1 a_1) (A - x_1 a_1^*) = \\ &= A^* A - x_1 A^* a_1^* - \bar{x}_1 a_1 A + |x_1|^2 a_1 a_1^* = \\ &= A A^* - x_1 (a_1^* A^* - \bar{x}_1) - \bar{x}_1 (A a_1 - x_1) + |x_1|^2 (a_1^* a_1 - 1) = \\ &= A A^* - x_1 a_1^* A^* - \bar{x}_1 A a_1 + |x_1|^2 a_1^* a_1 + |x_1|^2 = \\ &= (A - x_1 a_1^*) (A^* - \bar{x}_1 a_1) + |x_1|^2 = |x_1|^2 (1 + d_2 d_2^*). \end{aligned}$$

Here we use the evident fact that $AA^* = A^*A$.

The relation $d_1^*d_3 = d_3d_1^*$ follows immediately from the definition of d_3 and the commutation relations between d_1^* and d_2 , d_2^* . Using this commutation again as well as relations (13) we get

$$\begin{aligned} \sqrt{1 + \frac{|x_2|^2}{|x_1|^2}}(d_1d_3 - d_3d_1) &= d_1a_2 - a_2d_1 + |x_1|(d_1^*d_1 - d_1d_1^*)d_2 + \\ &+ \frac{x_1}{2}((d_1^*)^2d_1 - d_1(d_1^*)^2) = \\ &= -|x_1|d_2 - x_1d_1^* + |x_1|d_2 + \frac{x_1}{2}2d_1^* = 0, \end{aligned}$$

$$\begin{aligned} \sqrt{1 + \frac{|x_2|^2}{|x_1|^2}}(d_2^*d_3 - d_3d_2^*) &= d_2^*a_2 - a_2d_2^* - \\ &- \frac{\bar{x}_1}{2}(d_2^*d_2^2 - d_2^2d_2^*) - |x_1|(d_2^*d_2 - d_2d_2^*)d_1^* = \\ &= \bar{x}_1d_2 + |x_1|d_1^* - \frac{\bar{x}_1}{2}2d_2 - |x_1|d_1^* = 0 \end{aligned}$$

and

$$\begin{aligned} \sqrt{1 + \frac{|x_2|^2}{|x_1|^2}}(d_2d_3 - d_3d_2) &= d_2a_2 - a_2d_2 + \frac{x_2}{|x_1|}(d_2d_2^* - d_2^*d_2) = \\ &= \frac{x_2}{|x_1|} - \frac{x_2}{|x_1|} = 0. \end{aligned}$$

Finally, since $d_3d_i = d_id_3$, $d_i^*d_3 = d_3d_i^*$, $i = 1, 2$ one has

$$\begin{aligned} 1 = a_2^*a_2 - a_2a_2^* &= \left(1 + \frac{|x_2|^2}{|x_1|^2}\right)(d_3^*d_3 - d_3d_3^*) - \frac{|x_2|^2}{|x_1|^2}(d_2^*d_2 - d_2d_2^*) + \\ &+ \frac{|x_1|^2}{4}((d_2^*)^2d_2^2 - d_2^2(d_2^*)^2) + \frac{|x_1|^2}{4}(d_1^2(d_1^*)^2 - (d_1^*)^2d_1^2) + \\ &+ |x_1|^2(d_2^*d_2d_1d_1^* - d_1^*d_1d_2d_2^*) + \\ &+ \frac{\bar{x}_1|x_1|}{2}d_1(d_2^*d_2^2 - d_2^2d_2^*) + \frac{\bar{x}_1|x_1|}{2}d_2(d_1^2d_1^* - d_1^*d_1^2) + \\ &+ \frac{x_1|x_1|}{2}d_2^*(d_1(d_1^*)^2 - (d_1^*)^2d_1) + \frac{x_1|x_1|}{2}d_1^*((d_2^*)^2d_2 - d_2(d_2^*)^2) = \\ &= \left(1 + \frac{|x_2|^2}{|x_1|^2}\right)(d_3^*d_3 - d_3d_3^*) - \frac{|x_2|^2}{|x_1|^2} + \\ &+ \frac{|x_1|^2}{4}(2 + 4d_2d_2^*) - \frac{|x_1|^2}{4}(2 + 4d_1d_1^*) + |x_1|^2(d_1d_1^* - d_2d_2^*) + \\ &+ \frac{\bar{x}_1|x_1|}{2}2d_1d_2 - \frac{\bar{x}_1|x_1|}{2}2d_2d_1 - \frac{x_1|x_1|}{2}2d_2^*d_1^* + \frac{x_1|x_1|}{2}2d_1^*d_2^* = \\ &= \left(1 + \frac{|x_2|^2}{|x_1|^2}\right)(d_3^*d_3 - d_3d_3^*) - \frac{|x_2|^2}{|x_1|^2} \end{aligned}$$

showing that $d_3^*d_3 - d_3d_3^* = 1$. \square

Denote by A_3 the $*$ -algebra generated by CCR with 3 degrees of freedom and denote by c_1, c_2, c_3 the canonical generators of A_3 . Construct elements b_1, b_2, B of A_3 using formulas (15).

Lemma 8. *The elements b_1, b_2, B satisfy (11) and generate A_3 .*

Proof. It is evident that one can express $c_i, i = 1, 2, 3$ via b_1, b_2, B using (12) with b_1, b_2, B instead of a_1, a_2, A . So A_3 is generated by b_1, b_2, b_3 .

Let us show that b_1, b_2, B satisfy (11).

Indeed, it is a moment of reflection to see that $b_1^*b_2 = b_2b_1^*$ and $b_1^*b_1 - b_1b_1^* = 1$. Further

$$\begin{aligned} b_2b_1 - b_1b_2 &= |x_1|c_1^*c_2c_1 + \frac{x_1}{2}(c_1^*)^2c_1 - |x_1|c_1c_1^*c_2 - \frac{x_1}{2}c_1(c_1^*)^2 = \\ &= |x_1|(c_1^*c_1 - c_1c_1^*)c_2 + \frac{x_1}{2}((c_1^*)^2c_1 - c_1(c_1^*)^2) = \\ &= |x_1|c_2 + \frac{x_1}{2}2c_1^* = |x_1|c_2 + x_1c_1^* = B, \end{aligned}$$

$$\begin{aligned} Bb_1 &= |x_1|c_2c_1 + x_1c_1^*c_1 = |x_1|c_2c_1 + x_1(1 + c_1c_1^*) = \\ &= |x_1|c_1c_2 + x_1c_1c_1^* + x_1 = c_1(|x_1|c_2 + x_1c_1^*) + x_1 = b_1B + x_1, \end{aligned}$$

and

$$Bb_2 - b_2B = -\frac{x_2}{|x_1|}|x_1|c_2c_2^* + \frac{x_2}{|x_1|}x_1c_2^*c_2 = x_2(c_2^*c_2 - c_2c_2^*) = x_2.$$

Thus it remains to check that $b_2^*b_2 - b_2b_2^* = 1$. But in fact this was done in Lemma 7, when we checked that the relation $d_3^*d_3 - d_3d_3^* = 1$ is satisfied. \square

Using Lemma 7 and Lemma 8 it is easy to see that the $*$ -algebras A_{x_1, x_2} and A_3 are isomorphic.

Proposition 10. *The $*$ -algebra A_{x_1, x_2} is isomorphic to the $*$ -algebra A_3 .*

Proof. Let $\phi: A_{x_1, x_2} \rightarrow A_3$ be a homomorphism defined by

$$\phi(a_i) = b_i, \quad i = 1, 2, \quad \phi(A) = B,$$

where b_1, b_2 and B are the generators constructed in Lemma 8. Similarly define $\psi: A_3 \rightarrow A_{x_1, x_2}$ by

$$\psi(c_i) = d_i, \quad i = 1, 2, 3,$$

where d_i are taken from Lemma 7.

Then $\psi \circ \phi = \text{id}_{A_{x_1, x_2}}$ and $\phi \circ \psi = \text{id}_{A_3}$. \square

Therefore in order to study irreducible representations of A_{x_1, x_2} we can work with the generators d_1, d_2, d_3 . As for the case of representations annihilating \mathcal{K}_3 , we say that a representation of A_{x_1, x_2} with $x_1 \neq 0$ is *well-behaved* if the corresponding representation of $A_3 \simeq A_{x_1, x_2}$ is well-behaved. Then from the uniqueness of irreducible well-behaved $*$ -representation of CCR with finite degrees of freedom we get that the space of representation is $\mathcal{H} = l_2(\mathbb{Z}_+)^{\otimes 3}$ and

$$d_1 = a \otimes \mathbf{1} \otimes \mathbf{1}, \quad d_2 = \mathbf{1} \otimes a \otimes \mathbf{1}, \quad d_3 = \mathbf{1} \otimes \mathbf{1} \otimes a.$$

Returning to the generators a_1, a_2, a_3 using (12) we get the following result.

Theorem 2. *For any $(x_1, x_2) \in \mathbb{C}^2$ with $x_1 \neq 0$ there exists a unique, up to a unitary equivalence, well-behaved irreducible representation of A_{x_1, x_2} defined on the generators by the following formulas*

$$\begin{aligned} a_1 &= a \otimes \mathbf{1} \otimes \mathbf{1}, \\ a_2 &= \sqrt{1 + \frac{|x_2|^2}{|x_1|^2}} \mathbf{1} \otimes \mathbf{1} \otimes a - \frac{x_2}{|x_1|} \mathbf{1} \otimes a^* \otimes \mathbf{1} + \frac{\bar{x}_1}{2} \mathbf{1} \otimes a^2 \otimes \mathbf{1} + \\ &\quad + |x_1| a^* \otimes a \otimes \mathbf{1} + \frac{x_1}{2} (a^*)^2 \otimes \mathbf{1} \otimes \mathbf{1}, \\ A &= |x_1| \mathbf{1} \otimes a \otimes \mathbf{1} + x_1 a^* \otimes \mathbf{1} \otimes \mathbf{1}. \end{aligned}$$

5.2.2 Representations with $x_1 = 0$

Let $x_1 = 0$ and $x_2 \neq 0$. As in the previous case we have $A_{0, x_2} \simeq A_3$. To see this we express the generators a_1, a_2, a_3 via the generators d_1, d_2 and d_3 of CCR using formulas (15) with $a_2, -a_1$ instead of a_1, a_2 respectively, exchanging x_1 with x_2 and letting then $x_1 = 0$. For this we observe that $(-a_1)a_2 - a_2(-a_1) = A$, and $Aa_2 - a_2A = x_2$. Hence we get the following result.

Theorem 3. *For any $x_2 \in \mathbb{C}$, $x_2 \neq 0$, there exists a unique, up to a unitary equivalence, irreducible well-behaved $*$ -representation of A_{0, x_2} , defined by the following formulas*

$$\begin{aligned} a_2 &= a \otimes \mathbf{1} \otimes \mathbf{1}, \\ a_1 &= -(\mathbf{1} \otimes \mathbf{1} \otimes a + \frac{\bar{x}_2}{2} \mathbf{1} \otimes a^2 \otimes \mathbf{1} + \\ &\quad + |x_2| a^* \otimes a \otimes \mathbf{1} + \frac{x_2}{2} (a^*)^2 \otimes \mathbf{1} \otimes \mathbf{1}) \\ A &= |x_2| \mathbf{1} \otimes a \otimes \mathbf{1} + x_2 a^* \otimes \mathbf{1} \otimes \mathbf{1} \end{aligned}$$

If both $x_1 = 0$ and $x_2 = 0$, then $Aa_i = a_iA$, $i = 1, 2$ and hence the cubic ideal \mathcal{K}_3 is annihilated. In this case irreducible well-behaved representations are described in Theorem 1.

5.3 Concluding remarks

Note that our result shows in particular that in the case of \mathcal{A}_2^0 the ideal \mathcal{K}_4 does not coincide with \mathcal{I}_4 , the largest homogeneous ideal of degree 4. Indeed, as noted above $\mathcal{K}_2 \otimes \mathcal{K}_2 \subset \mathcal{I}_4$. So, if a representation π annihilates \mathcal{I}_4 , then $\pi(A^2) = 0$. Since A is a normal element we immediately have $\pi(A) = 0$. However the representation that we constructed above has the property that $\pi(\mathcal{K}_4) = \{0\}$ but $\pi(A) \neq 0$. Thus $\mathcal{K}_4 \neq \mathcal{I}_4$.

Acknowledgements

The work on this paper was supported by DFG grant SCHM1009/4-1. The paper was initiated during the visit of V. Ostrovskiy, D. Proskurin and L. Turowska to Leipzig University, the warm hospitality and stimulating atmosphere are gratefully acknowledged. We are also indebted deeply to D. Neiter for performing computations in MATHEMATICA.

References

- [1] M. Bożejko and R. Speicher, Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. *Mat. Ann.* **300** (1994), 97–120.
- [2] P.E.T. Jørgensen, D.P. Proskurin and Yu.S. Samoilenko, The kernel of Fock representation of Wick algebras with braided operator of coefficients, *Pacific J. Math.* **198** (2001), 109–122.
- [3] P.E.T.Jørgensen, L.M. Schmitt, and R.F.Werner, Positive representations of general commutation relations allowing Wick ordering. *J. Funct. Anal.* **134** (1995), 33–99.
- [4] W. Marcinek, On commutation relations for quons, *Rep. Math. Phys.*, **41** no. 2, (1998), pp. 155–172.
- [5] V. Ostrovskiy and Yu. Samoilenko, Introduction to the theory of representations of finitely presented algebras, *Rev. Math. & Math. Phys.*, (2000), **11**, Gordon & Breach, London, 261 p.
- [6] D. Proskurin, Homogeneous ideals in Wick \ast -algebras, *Proc. Amer. Math. Soc.*, **126** no. 11 (1998), pp. 3371–3376.
- [7] W. Pusz and S.L. Woronowicz, Twisted second quantization, *Rep. Math. Phys.*, **27** no. 2, (1989), pp. 231–257.
- [8] K.Schmüdgen, Unbounded operator algebras and representation theory, Birkhäuser Verlag, 1990.